A one-parameter deformation of the Farahat-Higman algebra

Jean-Paul Bultel

A ONE-PARAMETER DEFORMATION OF THE FARAHAT-HIGMAN ALGEBRA

JEAN-PAUL BULTEL

ABSTRACT. We show, by introducing an appropriate basis, that a one-parameter family of Hopf algebras introduced by Foissy [Adv. Math. 218 (2008) 136-162] interpolates beetween the Faà di Bruno algebra and the Farahat-Higman algebra. Its structure constants in this basis are deformation of the top connection coefficients, for which we obtain analogues of Macdonald's formulas.

1. INTRODUCTION

The center Z_n of the group algebra $\mathbb{Z}[\mathfrak{S}_n]$ of the symmetric group is spanned by conjugacy classes C_μ, which are parametrized by partitions μ of n. The *connection coefficients* $a_{\mu\nu}^{\lambda}$ are the structure constants

$$C_\mu C_\nu = \sum_{\lambda \vdash n} a_{\mu\nu}^{\lambda} C_\lambda \tag{1}$$

of Z_n. These coefficients, whose calculation is in general very hard, have important applications to various enumerative problems or to the calculation of certain matrix integrals [7].

For a partition $\mu = (\mu_1 \geq \mu_2 \geq \ldots \geq \mu_r > 0)$, define

$$\bar{\mu} = (\mu_1 - 1, \mu_2 - 1, \ldots, \mu_r - 1)\,, \tag{2}$$

the *reduced cycle type* of any permutation of cycle type μ. Denoting by $c_\rho(n)$ the conjugacy class C_μ of $\mathfrak{S}_n$ such that $\bar{\mu} = \rho$, we can rewrite (1) in the form

$$c_\mu(n) c_\nu(n) = \sum_{\lambda} a_{\mu\nu}^{\lambda}(n) c_\lambda(n)\,. \tag{3}$$

It has been proved by Farahat and Higman [4] that the connection coefficients $a_{\mu\nu}^{\lambda}(n)$ are polynomial functions of n, and are independent of n if $|\lambda| = |\mu| + |\nu|$. These are called the *top connection coefficients*.

One may use the top connection coefficients to define an algebra R, spanned by formal symbols c_μ indexed by all partitions, whith multiplication rule

$$c_\mu c_\nu = \sum_{|\lambda| = |\mu| + |\nu|} a_{\mu\nu}^{\lambda} c_\lambda\,. \tag{4}$$

This is the Farahat-Higman algebra ([4], see also [10, ex. 24 p 131]). It is also proved in [4] that R is isomorphic to the algebra of symmetric functions Λ. The construction of an explicit isomorphism $\varphi :\ \Lambda \to R$ is more recent, and due to Macdonald [10, ex.

Date: June 7, 2010.

25 p 132] (another proof has been given by Goulden and Jackson [6]). More precisely, Macdonald constructed a basis (g_μ) of Λ such that

$$g_\mu g_\nu = \sum_{|\lambda|=|\mu|+|\nu|} a^\lambda_{\mu\nu} g_\lambda, \tag{5}$$

and obtained a recursive formula for the calculation of $a^\lambda_{\mu\nu}$.

Murray [12] explicited the images in Λ of the projections of symmetric functions of Jucys-Murphy elements in R under this isomorphism. In Section 3, we give a new derivation of his result, and new proofs of various results of Biane [1] and Matsumoto-Novak [11].

It is well-known that the algebra of symmetric functions is a Hopf algebra. Its standard coproduct, denoted here by Δ_0, comes from its interpretation as the algebra of polynomial functions on the multiplicative group G_0 of formal power series with constant term 1. It has, however, another Hopf algebra structure, known as the *Faà di Bruno algebra*, coming from its interpretation as the algebra of polynomial functions on the group $G_1 = \{ta(t)|a \in G_0\}$ of formal diffeomorphisms of the line under composition (see, *e.g.*, [3]). In fact, Macdonald's basis g_μ is the dual of the image $h_\mu = S_1(h_\mu)$ of the basis of complete homogeneous functions h_μ by the antipode of the Faà di Bruno algebra $\mathcal{F}$.

This suggests to interpret g_μ as living in the dual $\mathcal{F}^*$ of $\mathcal{F}$. However, contrary to R, this dual is not commutative, being the universal enveloping algebra of the Lie algebra $\mathfrak{g}_1$ of G_1. Thus, such an interpretation does not make sense *a priori*.

To clarify this situation, we make use of a one-parameter deformation $\mathcal{F}_\gamma$ of $\mathcal{F}$ recently discovered by Foissy [5] in his investigation of combinatorial Dyson-Schwinger equations in the Connes-Kreimer algebra. We then obtain for the structure constants of $\mathcal{F}_\gamma$ in the dual basis of $S_1(h_\mu)$ a one-parameter deformation of Macdonald's formulas which are recovered for $\gamma = 0$.

We follow the conventions of [10]. For the convenience of the reader, the most essential ones are recalled in Section 2.

2. Notations and background

2.1. **Partitions.** Let n be a positive integer. A finite sequence of strictly positive integers $(\lambda_1, \lambda_2, \ldots)$ is called a *partition* of n if $\lambda_1 \geq \lambda_2 \geq \ldots$ and $\lambda_1 + \lambda_2 + \ldots = n$. We then write $\lambda \vdash n$. The λ_i are the *parts* of λ, $|\lambda| = n$ is the *weight* of λ and the number $l(\lambda)$ of parts in λ is the *length* of λ. The *multiplicity* $m_\lambda(k)$ of k in λ is the number of parts in λ equal to k. We set

$$z_\lambda = \prod_{k\geq 1} k^{m_k(\lambda)} m_k(\lambda)! \tag{6}$$

2.2. **The algebra of symmetric functions.** We denote by Λ the algebra of symmetric functions. The bases (m_λ),(e_λ), (h_λ),(p_λ) and (s_λ) are respectively the *monomial*, *elementary*, *complete*, *power sum* and *Schur* symmetric functions. These bases of Λ are parametred by partitions of all integers. For any basis (b_λ), we denote by b_n

the symmetric function $b_{(n)}$. Denote by $\langle\ ,\ \rangle$ the usual *scalar product* on Λ, for which (s_λ) is an orthonormal basis. For this scalar product, (p_λ) is an orthogonal basis, and one has

$$\langle p_\lambda, p_\mu\rangle = \delta_{\lambda\mu} z_\mu \tag{7}$$

The bases (m_λ) and (h_λ) are dual to each other.

$$\langle h_\lambda, m_\mu\rangle = \delta_{\lambda\mu} \tag{8}$$

(δ is the Kronecker symbol). We denote by $p_n^\perp$ the *adjoint* of the multiplication by p_n, in the sense

$$\langle p_n f, g\rangle = \langle f, p_n^\perp(g)\rangle \tag{9}$$

where f and g are any two symmetric functions. This operator is a *derivation.* More precisely,

$$p_n^\perp = n\frac{\partial}{\partial p_n} \tag{10}$$

This operator acts on the h_n as follows

$$p_n^\perp h_N = h_{N-n} \tag{11}$$

for any $N > n$, so that

$$p_n^\perp = \sum_{r\geq 0} h_r \frac{\partial}{\partial h_{n+r}} \tag{12}$$

Now, let X and Y be two alphabets. For any symmetric function f and any scalar k, we identify f with $f(X)$, and we define the algebra morphisms $f \to f(X+Y)$, $f \to f(kX)$ and $f \to f(k)$ by

$$p_n(X+Y) = p_n(X) + p_n(Y) \tag{13}$$

$$p_n(kX) = kp_n(X) \tag{14}$$

$$p_n(k) = k \tag{15}$$

The standard Hopf algebra structure of Λ is defined by the coproduct Δ_0

$$\Delta_0(f) = f(X+Y) \tag{16}$$

where we identify $f(X+Y)$ with an element of $\Lambda\otimes\Lambda$ by identifying $f\otimes g$ with $f(X)g(Y)$. The counit ϵ and the antipode S_0 are given by

$$\epsilon(f) = f(0) \tag{17}$$

and

$$S_0(f) = f(-X) \tag{18}$$

Denote by $\mathcal{H}_0$ this Hopf algebra.

$$\mathcal{H}_0 = (\Lambda, ., 1, \Delta_0, \epsilon, S_0) \tag{19}$$

One can give another interpretation of $\mathcal{H}_0$. Let

$$G_0 = \{a \mid a(t) = 1 + a_1 t + a_2 t^2 + \ldots\} = 1 + t\mathbb{C}[[t]] \tag{20}$$

be the multiplicative group of formal power series with constant term 1, and let $\mathcal{H}$ be the algebra of polynomial functions on G_0. Let $k_n \in \mathcal{H}$ be the map defined by

$$k_n(a) = a_n \tag{21}$$

The standard coproduct for functions on a group is

$$\Delta f(a,b) = f(ab) \tag{22}$$

where Δf is identified with an element of $\mathcal{H} \otimes \mathcal{H}$. Then, $h_n \mapsto k_n$ is an isomorphism of Hopf algebras $\mathcal{H}_0 \to \mathcal{H}$.

2.3. **Other bases of the algebra of symmetric functions.** The formal series

$$u = tH(t) = t + h_1 t^2 + h_2 t^3 + \dots \tag{23}$$

has an inverse for the composition, so that we can rewrite t in terms of u, as follows.

$$t = u + h_1^\star u^2 + h_2^\star u^3 + \dots \tag{24}$$

The $h_k^\star$ are homogeneous symmetric functions of degree k, and the algebra morphism ψ from Λ to Λ defined by $\psi(h_k) = h_k^\star$ is an *involution* :

$$\psi^2 = Id_\Lambda \tag{25}$$

The Lagrange inversion formula shows that $(n+1)h_n^\star$ is the coefficient of t^n in $H(t)^{-(n+1)}$, so that

$$h_n^\star = \frac{h_n(-(n+1)X)}{n+1} \tag{26}$$

We can define a multiplicative $\mathbb{Z}$-basis $(h_\lambda^\star)$ of Λ, by

$$h_\lambda^\star = h_{\lambda_1}^\star h_{\lambda_2}^\star \dots = \psi(h_\lambda) \tag{27}$$

In [10, ex. 24 p 35], Macdonald shows that

$$(n+1)h_n^\star = \sum_{\lambda \vdash n} (-1)^{l(\lambda)} \binom{n+l(\lambda)}{n} u_\lambda h_\lambda \tag{28}$$

where

$$u_\lambda = \frac{l(\lambda)}{\prod_{i \geq 1} m_i(\lambda)!} \tag{29}$$

Let (g_λ) be the adjoint basis of $(h_\lambda^\star)$, that is

$$\langle g_\lambda, h_\mu^\star \rangle = \delta_{\lambda\mu} \tag{30}$$

One has

$$g_n = -m_n = -p_n \tag{31}$$

The algebra morphism ψ^T (adjoint of ψ) is also an involution, and it maps g_λ to m_λ. The matrix Ψ of ψ is *strictly upper triangular* in the basis (h_λ) and one has

$$\Psi_{\lambda\lambda} = (-1)^{l(\lambda)} \tag{32}$$

Theorem 2.1 (Macdonald [10]). *The linear map*

$$\Phi : \left\{ \begin{array}{ccc} R & \rightarrow & \Lambda \\ c_\lambda & \mapsto & g_\lambda \end{array} \right. \tag{33}$$

is an isomorphism of algebras.

2.4. **Top connection coefficients.** Now, let us recall some properties of the top connection coefficients $a^\nu_{\lambda\mu}$, which are the structure constants of the Farahat-Higman algebra R. By Theorem 2.1,

$$g_\lambda g_\mu = \sum_{|\nu|=|\lambda|+|\mu|} a^\nu_{\lambda\mu} g_\nu \tag{34}$$

When μ and ν are partitions of length 1 ($\mu = (r)$ and $\nu = (m)$), Macdonald gives in [10, ex. 24 p 131] an explicit formula for $a^{(m)}_{\lambda(r)}$, $m = |\lambda| + r$, that is

$$a^{(m)}_{\lambda(r)} = \left\{ \begin{array}{ll} \frac{(m+1)r!}{(r+1-l(\lambda))\prod_{i>0} m_i(\lambda)!} & \text{if } l(\lambda) \leq r+1 \\ \\ 0 & \text{otherwise} \end{array} \right. \tag{35}$$

Moreover, Macdonald gives a recurrence formula, for any partition ν with $|\nu| = |\lambda|+r$

$$a^\nu_{\lambda(r)} = \sum_{(i,\mu)/\mu\cup\nu=\lambda\cup(\nu_i)} a^{(\nu_i)}_{\mu(r)} \tag{36}$$

One can deduce from these formulas that the $a^\nu_{\lambda r}$ are zero except if $\nu \geq \lambda \cup (r)$, and that $a^{\lambda\cup(r)}_{\lambda(r)} > 0$. The multiplicative structure of the Farahat-Higman algebra is uniquely determined by (35) and (36).

2.5. **Jucys-Murphy elements.** Let n and i be two integers, $i \leq n$. Define

$$\xi_i = \sum_{j<i} (j, i) \in \mathbb{C}[\mathfrak{S}_n], \tag{37}$$

called the *ith Jucys-Murphy element*. It is the sum of all transpositions (i, j) with $j < i$. The Jucys-Murphy elements do not belong to the center Z_n of $\mathbb{C}[\mathfrak{S}_n]$, but they generate a maximal commutative subalgebra of $\mathbb{C}[\mathfrak{S}_n]$. Let

$$\Xi_n = \{\xi_1, \xi_2, \ldots, \xi_n\} \tag{38}$$

be the alphabet of Jucys-Murphy elements. It is known that the algebra of symmetric functions in Ξ_n is exactly Z_n [8]. Hence, since $(C_\mu)_{\mu\vdash n}$ is a basis of Z_n, one can rewrite $f(\Xi_n)$ as

$$f(\Xi_n) = \sum_{\mu\vdash n} k_{f,\mu}(n) C_\mu \tag{39}$$

for any homogeneous symmetric function f, and this decomposition is unique. We can rewrite this formula in terms of the reduced cycle types

$$f(\Xi_n) = \sum k_{f,\mu}(n) c_\mu(n). \tag{40}$$

Let r be the degree of f. When $|\mu| = r$, $k_{f,\mu}(n)$ does not depend on n. Let us rewrite it as $k_{f,\mu}$ and define a new element $f(\Xi)$ of the Farahat-Higman algebra R, by considering only maximal terms in the expansion of $f(\Xi_n)$ for n large enough.

$$f(\Xi) = \sum_{\mu \vdash r} k_{f,\mu} c_\mu \tag{41}$$

Murray [12] has shown that for any f, the isomorphism Φ between R and Λ maps $f(\Xi)$ to $f(-X) = S_0(f)$.

$$\Phi(f(\Xi)) = f(-X) \tag{42}$$

Since $\Phi(c_\mu) = g_\mu$, using the duality between the bases $(h^\star_\lambda)$ and (g_λ), one has

$$k_{f,\mu} = \langle f(-X), h^\star_\mu \rangle \tag{43}$$

3. More about Murray's result

3.1. **A new derivation.** Let us give a new proof of (42). Lascoux and Thibon [9] give the formula

$$p_m(\Xi_n) = \sum_{k=1}^{m+1} \sum_{\kappa \vdash k, l(\kappa) \le m-k+2} \phi_{\kappa,m} a_{\kappa,n} \tag{44}$$

where $a_{\kappa,n}$ is defined by

$$a_{\kappa,n} = \frac{1}{(n-k)!} z_{\kappa \cup 1^{n-k}} C_{\kappa \cup 1^{n-k}} \tag{45}$$

The C_λ are the classical conjugacy classes, and the $\phi_{\kappa,m}$ are defined from

$$\phi_\kappa(t) = \frac{(1-q^{-1})^{k-1}}{k! z_k} \frac{p_\kappa(q-1)}{q-1}\bigg|_{q=e^t} \tag{46}$$

by

$$\phi_\kappa(t) = \sum_{m \ge |\kappa| + l(\kappa) - 2} \phi_{\kappa,m} \frac{t^m}{m!}. \tag{47}$$

From (44), $C_{(m+1)\cup 1^{n-k}}$ is the only class of modified weight m which can give a contribution to $p_m(\Xi_n)$. Indeed, for $|\kappa| - l(\kappa) = m$, if we had $l(\kappa) > 1$ we would have $|\kappa| > m+1$, but the κ satisfying this inequality do not occur in the sum (44), so that the formula

$$p_m(\Xi) = \sum_{\mu \vdash m} k_\mu c_\mu \tag{48}$$

becomes

$$p_m(\Xi) = k_m c_m \tag{49}$$

Now we only have to determine the coefficient k_m. In order to do that, suppose that $n = m+1$. In this case we have

$$p_m(\Xi_{m+1}) = \xi_1^m + \xi_2^m + \ldots + \xi_{m+1}^m, \tag{50}$$

that is,

$$p_m(\Xi_{m+1}) = 0^m + (1,2)^m + ((1,3)+(2,3))^m + \dots \\ +((1,m+1)+(2,m+1)+\dots+(m,m+1))^m \tag{51}$$

Only the last term can give a contribution to C_{m+1}. This term can be rewritten

$$\sum (i_1, m+1)(i_2, m+1)\dots(i_m, m+1) \tag{52}$$

where the sum is over the $(i_1, i_2, \dots, i_m)$ where the i_k are integers such that $1 \leq i_k \leq m$. In this new sum, only terms with the i_k all distinct can give a contribution to C_{m+1}. The number of these terms is $m!$, that is, the cardinality of C_{m+1}, so that the coefficient of C_{m+1} in $p_m(\Xi_{m+1})$ is 1. This coefficient does not depend on n since $(m+1)$ has a maximal modified weight, so that the coefficient of c_m in $p_m(\Xi)$, that is k_m, is again 1, so that

$$p_m(\Xi) = c_m. \tag{53}$$

Hence, $\Phi(p_m(\Xi)) = \Phi(c_m) = g_m = -p_m$, and

$$\Phi(p_\lambda(\Xi)) = \Phi(\prod_i p_{\lambda_i}(\Xi)) = \prod_i \Phi(p_{\lambda_i}(\Xi)). \tag{54}$$

From that we have $\Phi(p_\lambda(\Xi)) = \prod_i(-p_{\lambda_i}) = (-1)^{l(\lambda)} p_\lambda = p_\lambda(-X)$, and since Λ is spanned by the p_μ, this implies (42).

3.2. **Examples.** Using his result and simple calculations in Λ, Murray [12] computes coefficients in the expansion of certain symmetric functions of the Jucys-Murphy elements over the c_λ. For example, he shows the following formulas

$$\langle e_k(-X), h_\lambda^\star \rangle = 1 \tag{55}$$

$$\langle h_k(-X), h_\lambda^\star \rangle = \prod_i \mathrm{Cat}_{\lambda_i - 1} \tag{56}$$

(where Cat_i is the ith Catalan number) giving the top coefficients in the expansion of $e_k(\Xi)$ and $h_k(\Xi)$. In the same vein, let us give a new proof of a result of Matsumoto and Novak for the monomial functions. Let k be an integer, $\lambda \vdash k$ and $\mu \vdash k$. We denote by L_μ^λ the coefficient defined by

$$m_\lambda(\Xi) = \sum_{|\mu| = |\lambda|} L_\mu^\lambda c_\mu \tag{57}$$

Matsumoto and Novak [11] show that

$$L_\mu^\lambda = \sum_{(\lambda^{(1)}, \lambda^{(2)}, \dots) \in \mathfrak{R}(\lambda,\mu)} RC(\lambda^{(1)}) RC(\lambda^{(2)}) \dots \tag{58}$$

where

$$\mathfrak{R}(\lambda,\mu) = \{(\lambda^{(1)}, \lambda^{(2)}, \dots) / \forall i, \lambda^{(i)} \vdash \mu_i \text{ and } \lambda = \lambda^{(1)} \cup \lambda^{(2)} \cup \dots\} \tag{59}$$

and for any partition λ of N,

$$RC(\lambda)=\frac{1}{N+1}m_\lambda(N+1)=\frac{|\lambda|!}{(|\lambda|-l(\lambda)+1)!\prod_{i\geq 1}m_i(\lambda)!} \tag{60}$$

Matsumoto and Novak also give a combinatorial interpretation for $RC(\lambda)$. Let us show how to derive (58) from (43). We have

$$\begin{aligned} L_\mu^\lambda &= \langle m_\lambda(-X), h_\mu^*\rangle \\ &= \langle m_\lambda, h_\mu^*(-X)\rangle \\ &= \left\langle m_\lambda, \frac{h_{\mu_1}((\mu_1+1)X)h_{\mu_2}((\mu_2+1)X)\ldots}{(\mu_1+1)(\mu_2+1)\ldots}\right\rangle \\ &= \frac{1}{\prod_i(\mu_i+1)}\left\langle m_\lambda, \prod_i \sum_{\lambda^{(i)}\vdash\mu_i} m_{\lambda^{(i)}}(\mu_i+1)h_{\lambda^{(i)}}\right\rangle \end{aligned} \tag{61}$$

Expanding the right factor of the scalar product, we obtain nonzero terms only if $h_{\lambda^{(1)}}h_{\lambda^{(2)}}\ldots=h_\lambda$, since otherwise one has $\langle m_\lambda, h_{\lambda^{(1)}}h_{\lambda^{(2)}}\ldots\rangle=0$. Hence,

$$L_\mu^\lambda=\frac{1}{\prod_i(\mu_i+1)}\sum(m_{\lambda^{(1)}}(\mu_1+1))(m_{\lambda^{(2)}}(\mu_2+1))\ldots\langle m_\lambda,h_\lambda\rangle \tag{62}$$

The sum is over the $(\lambda^{(1)},\lambda^{(2)},\ldots)$ such that $\lambda^{(i)}\vdash\mu_i$ for all i and $\bigcup_i\lambda^{(i)}=\lambda$, that is, over the set $\mathfrak{R}(\lambda,\mu)$. Moreover, one has $\langle m_\lambda,h_\lambda\rangle=1$, so that

$$L_\mu^\lambda=\sum_{(\lambda^{(1)},\lambda^{(2)},\ldots)\in\mathfrak{R}(\lambda,\mu)}\frac{m_{\lambda^{(1)}}(\mu_1+1)}{\mu_1+1}\frac{m_{\lambda^{(2)}}(\mu_2+1)}{\mu_2+1}\ldots \tag{63}$$

This is the result of Matsumoto and Novak.

3.3. **Coefficient of the cycle with maximal length in $p_\lambda(\Xi_n)$.** We call *modified cycle type* of a product of cycles the sequence $(l_1-1,l_2-1,\ldots)$, where the l_i are the lengths of the factors. For example, the product (13)(234) has modified cycle type $(2-1,3-1)=(1,2)$. Biane [1] obtains an explicit formula for the number α_λ of factorisations with modified cycle type λ for a cycle of length $n+1$, where $n=|\lambda|=\sum_i\lambda_i$, that is

$$\alpha_\lambda=(n+1)^{l(\lambda)-1} \tag{64}$$

Let us prove this with symmetric functions. We consider in the Farahat-Higman algebra R the product

$$c^\lambda=c_{\lambda_1}c_{\lambda_2}\ldots \tag{65}$$

Since R is isomorphic to Λ, it is commutative, so that the order of the elements of λ has no importance. Hence, we can assume that λ is a partition. One has

$$\begin{aligned}\Phi(c^\lambda) &= g_{\lambda_1}g_{\lambda_2}\ldots=(-p_{\lambda_1})(-p_{\lambda_2})\ldots \\ &= (-1)^{l(\lambda)}p_\lambda=p_\lambda(-X)\end{aligned} \tag{66}$$

Hence, from (42),

$$c^\lambda=p_\lambda(\Xi) \tag{67}$$

Moreover, c^λ is the sum of all cycle products of modified type λ, and c_n is the sum of all the cycles of length $n+1$. Hence, the total number of cycle products of modified type λ is k_λ^n card(c_n), where k_λ^n is the coefficient of c_n in c^λ. The number α_λ of these factorisations does not depend on the choice of the cycle, so that $\alpha_\lambda = k_n^\lambda \frac{\mathrm{card}(c_n)}{\mathrm{card}(c_n)} = k_\lambda^n$, which is also from (67) the coefficient of c_n in $p_\lambda(\Xi)$. Hence, one has

(68) $$\alpha_\lambda = (-1)^{l(\lambda)}\langle p_\lambda, h_n^*\rangle,$$

so that

(69) $$\alpha_\lambda = \frac{1}{n+1}(n+1)^{l(\lambda)}\langle p_\lambda, h_n\rangle = (n+1)^{l(\lambda)-1}\langle p_\lambda, h_n\rangle$$

(since the bases (m_λ) and (h_λ) are dual to each other, $\langle p_\lambda, h_n\rangle$ is the coefficient of m_n in p_λ, that is, 1).

3.4. **Generalization.** Here, we give a combinatorial interpretation for the coefficient of c_μ in $p_\lambda(\Xi)$. Now we have

(70) $$\langle p_\lambda, h_n^\star\rangle = (-1)^{l(\lambda)}(n+1)^{l(\lambda)-1}$$

On another hand, since $p_\lambda(\Xi) = c_{\lambda_1}c_{\lambda_2}\ldots$ in R,

(71) $$c_{\lambda_1}c_{\lambda_2}\ldots = \sum_{|\lambda|=|\mu|}\langle p_\lambda(-X), h_\mu^*\rangle c_\mu,$$

so that $\langle p_\lambda(-X), h_\mu^*\rangle$ corresponds to the total number of decompositions in an arbitrary product of modified type λ of a product of disjoint cycles with modified type μ. In such a decomposition, each c_{μ_i} comes from certain $\prod_i c_{\lambda^{(i)}}$, where $\lambda^{(i)} \vdash \mu_i$ is a subpartition of λ, with $\lambda^{(1)} \cup \lambda^{(2)} \cup \ldots = \lambda$, that is, $(\lambda^{(1)}, \lambda^{(2)}, \ldots) \in \mathfrak{R}(\lambda,\mu)$. Moreover, each $(\lambda^{(1)}, \lambda^{(2)}, \ldots)$ must be counted $m(\lambda^{(1)}, \lambda^{(2)}, \ldots)$ times, where

(72) $$m(\lambda^{(1)}, \lambda^{(2)}, \ldots) = \prod_{j\geq 1}\frac{m_j(\lambda)!}{m_j(\lambda^{(1)})!m_j(\lambda^{(2)})!\ldots}$$

For a given c_{μ_i}, the number of decompositions of c_{μ_i} of a certain type $\prod_i c_{\lambda^{(i)}}$ corresponds to the coefficient of c_{μ_i} in $p_{\lambda^{(i)}}(\Xi)$, that is $\langle p_{\lambda^{(i)}}(-X), h_{\mu_i}^*\rangle$, so that we have from (70)

(73) $$\langle p_\lambda(-X), h_\mu^*\rangle = \sum_{(\lambda^{(1)},\lambda^{(2)},\ldots)\in\mathfrak{R}(\lambda,\mu)} m(\lambda^{(1)}, \lambda^{(2)}, \ldots)\prod_i(\mu_i+1)^{l(\lambda^{(i)})-1}$$

Note that it is nonzero only if λ is a refinement of μ. Finally, one has

(74)
$$p_\lambda(\Xi) = \sum_{|\lambda|=|\mu|}\left(\sum_{(\lambda^{(1)},\lambda^{(2)},\ldots)\in\mathfrak{R}(\lambda,\mu)}\left(\prod_j\frac{m_j(\lambda)!}{m_j(\lambda^{(1)})!m_j(\lambda^{(2)})!\ldots}\right)\left(\prod_i(\mu_i+1)^{l(\lambda^{(i)})-1}\right)\right)c_\mu$$

Note that from (70) we have

(75) $$h_n^\star = \sum_{\mu\vdash n}(-1)^{l(\mu)}(n+1)^{l(\mu)-1}\frac{p_\mu}{z_\mu}$$

Expanding $h^\star_\mu$ by means of this expression, since we have for $\bigcup_i \lambda^{(i)} = \lambda$

$$\prod_{j\geq 1} \frac{m_j(\lambda)!}{m_j(\lambda^{(1)})!m_j(\lambda^{(2)})!\dots} = \frac{z_\lambda}{z_{\lambda^{(1)}} z_{\lambda^{(2)}}\dots} \tag{76}$$

we see that (73) can also be obtained by simple calculations in Λ.

4. The Faà di Bruno algebra and its deformation

4.1. **The Faà di Bruno algebra.** There is another coproduct on Λ, denoted here by Δ_1 and defined by

$$\Delta_1 h_n = \sum_{k=0}^{n} h_k(X) \otimes h_{n-k}((k+1)X) \tag{77}$$

or equivalently

$$\Delta_1 h_n = \sum_{k=0}^{n} \sum_{\mu\vdash n-k} m_\mu(k+1)\ h_k \otimes h_\mu \tag{78}$$

This coproduct defines a Hopf algebra with the counit ϵ as in $\mathcal{H}_0$, and the antipode $S_1 = \psi$, that is, the involution mapping h_λ to h^*_λ. The Hopf algebra

$$\mathcal{H}_1 = (\Lambda, ., 1, \Delta_1, \epsilon, S_1) \tag{79}$$

is called the *Faà di Bruno algebra.* It has the following interpretation. Let

$$G_1 = \{\alpha \mid \exists a \in G_0, \forall t, \alpha(t) = ta(t)\} = tG_0 = t + t^2\mathbb{C}[[t]] \tag{80}$$

be the group of formal diffeomorphisms of the line tangent to the identity, and let again k_n be the linear form

$$k_n : \alpha(t) = t + a_1t^2 + a_2t^3 + \dots \mapsto a_n \tag{81}$$

Let Δ be the coproduct such that $\Delta k_n(\alpha, \beta)$ is the coefficient of t^{n+1} in

$$(\alpha \circ \beta)(t) = \alpha(\beta(t)) \tag{82}$$

The bialgebra $\mathcal{F}$ defined by this coproduct is isomorphic to $\mathcal{H}_1$ under the correspondence $k_n \mapsto h_n$.

4.2. **A deformation of the Faà di Bruno algebra.** Let γ be a real parameter in $[0, 1]$, and Δ_γ be the coproduct on Λ defined by

$$\Delta_\gamma(h_n) = \sum_{k=0}^{n} h_k \otimes h_{n-k}((k\gamma + 1)X) \tag{83}$$

or equivalently

$$\Delta_\gamma(h_n) = \sum_{k=0}^{n} \sum_{\mu\vdash n-k} m_\mu(k\gamma + 1)h_k \otimes h_\mu \tag{84}$$

Foissy obtains this coproduct in [5] in his investigation of formal Dyson-Schwinger equations in the Connes-Kreimer Hopf algebra, and shows that for $\gamma \in]0, 1]$, the resulting bialgebras $\mathcal{H}_\gamma$ are Hopf algebras, isomorphic to the Faà di Bruno algebra. Obviously, Δ_0 corresponds to $\mathcal{H}_0$.

4.3. **Graded dual of the deformed Fàa di Bruno algebra.** Consider now the Hopf algebra $\mathcal{H}'_\gamma$, the graded dual of the deformation $\mathcal{H}_\gamma$ considered in section 3. Denote by $\star_\gamma$ the product on $\mathcal{H}'_\gamma$. For any $f \in \mathcal{H}'_\gamma$ and any symmetric function g in $\mathcal{H}_\gamma$, denote by $\langle f, g\rangle$ the action of f on g. One has for any $f \in \mathcal{H}'_\gamma$, $g \in \mathcal{H}'_\gamma$ and $h \in \Lambda$, by definition of a dual Hopf algebra,

$$\langle f \star_\gamma g, h\rangle = \langle f \otimes g, \Delta_\gamma(h)\rangle \tag{85}$$

where

$$\langle a \otimes b, c \otimes d\rangle = \langle a, c\rangle\langle b, d\rangle \tag{86}$$

Denote by (d_λ), (q_λ) and (b_λ) the bases respectively dual to (h_λ), $\left(\frac{p_\lambda}{z_\lambda}\right)$ and h^*_λ in the sense of $\mathcal{H}'_\gamma$, that is, for example

$$\langle d_\lambda, h_\mu\rangle = \delta_{\lambda\mu} \tag{87}$$

For any γ in $[0, 1]$,

$$q_n = d_n = -b_n \tag{88}$$

Note that these b_n *generate* $\mathcal{H}'_\gamma$. $\mathcal{H}_0$ is *self-dual*, so that in the case $\gamma = 0$,

$$d_\lambda = m_\lambda, \ , q_\lambda = p_\lambda, \ , b_\lambda = g_\lambda \tag{89}$$

When $\gamma \neq 0$, $\mathcal{H}_\gamma$ is not self-dual and not commutative, since the coproduct Δ_γ on Λ is not cocommutative.

4.4. **Multiplicative structure of $\mathcal{H}'_\gamma$.** Now, let f and g be two elements of $\mathcal{H}'_\gamma$. One has for any partition μ,

$$\begin{aligned}
\langle f \star_\gamma g, h_\mu\rangle &= \langle f \otimes g, \Delta_\gamma(h_\mu)\rangle \\
&= \langle f \otimes g, \Delta_\gamma(h_{\mu_1})\Delta_\gamma(h_{\mu_2})\dots\rangle \\
&= \left\langle f \otimes g, \prod_i \sum_{k_i+l_i=\mu_i} h_{k_i} \otimes h_{l_i}((\gamma k_i + 1)X))\right\rangle \\
&= \left\langle f \otimes g, \sum_{k_i+l_i=\mu_i} \prod_i h_{k_i} \otimes h_{l_i}((\gamma k_i + 1)X)\right\rangle \\
&= \sum \langle f \otimes g, \prod_i h_{k_i} \otimes h_{l_i}((\gamma k_i + 1)X)\rangle
\end{aligned} \tag{90}$$

where the parameters of the sum are the same as above. Hence,

$$\begin{aligned}
\langle f \star_\gamma g, h_\mu\rangle &= \sum \langle f, h_{(k_1,k_2,\dots)}\rangle\langle g, \prod_i h_{l_i}(\gamma k_i + 1)X)\rangle \\
&= \sum \langle f, h_{(k_1,k_2,\dots)}\rangle\langle g, \prod_i \sum_{\rho\vdash l_i} m_\rho(\gamma k_i + 1)h_\rho\rangle
\end{aligned} \tag{91}$$

Now let us change the parameters of the sum, considering the partitions ρ_i, with $|\rho_i| = l_i$ and $k_i = \mu_i - |\rho_i|$. The sum is now over the $(\rho_1, \rho_2, \dots, \rho)$, such that for all i, $|\rho_i| \leq \mu_i$, and ρ is the union of the ρ_i. We obtain

$$\langle f \star_\gamma g, h_\mu \rangle = \sum \langle f, h_{(\mu_1 - |\rho_1|, \mu_2 - |\rho_2|, \dots)} \rangle \langle g, \prod_i m_{\rho_i}(\gamma\mu_i - \gamma|\rho_i| + 1) h_{\rho_i} \rangle, \tag{92}$$

and finally,

$$\langle f \star_\gamma g, h_\mu \rangle = \sum (\prod_i m_{\rho_i}(\gamma\mu_i - \gamma|\rho_i| + 1)) \langle f, h_{(\mu_1 - |\rho_1|, \mu_2 - |\rho_2|, \dots)} \rangle \langle g, h_\rho \rangle \tag{93}$$

One can use this formula to expand $f \star_\gamma g$ on the d_μ, where f and g are any two elements in $\mathcal{H}'_\gamma$

4.5. **Action of $q_n^\perp$ on the h_μ.** We define an operator $q_n^\perp$ on Λ as follows, for $f \in \mathcal{H}'_\gamma$ and $g \in \mathcal{H}_\gamma$.

$$\langle f \star_\gamma q_n, g \rangle = \langle f, q_n^\perp g \rangle \tag{94}$$

The operator $q_n^\perp$ is a *derivation*, because it is the adjoint of the right multiplication by a *primitive element*:

$$q_n^\perp(fg) = f q_n^\perp(g) + q_n^\perp(f) g \tag{95}$$

We shall need its action on the h_μ. Let $n > 0$ and μ be a partition. We can write from (93) :

$$\langle f \star_\gamma q_n, h_\mu \rangle = \sum (\prod_i m_{\rho_i}(\gamma\mu_i - \gamma|\rho_i| + 1)) \langle f, h_{(\mu_1 - |\rho_1|, \mu_2 - |\rho_2|, \dots)} \rangle \langle q_n, h_\rho \rangle \tag{96}$$

Since $q_n = d_n$ for all n, one has $\langle q_n, h_\rho \rangle = \delta_{n\rho}$, so that a term in this sum gives a nonzero contribution only if

$$\rho = (n) \tag{97}$$

In this case we have also $\langle q_n, h_\rho \rangle = 1$, and

$$\langle f \star_\gamma q_n, h_\mu \rangle = \sum_{i/\mu_i \geq n} m_n(\gamma\mu_i - \gamma n + 1) \langle f, h_{\mu \setminus \mu_i \cup (\mu_i - n)} \rangle \tag{98}$$

Since $m_n(\gamma\mu_i - \gamma n + 1) = \gamma(\mu_i - n) + 1$, one can deduce

$$q_n^\perp h_\mu = \sum_{i/\mu_i \geq n} (\gamma\mu_i - \gamma n + 1) h_{\mu \setminus \mu_i \cup (\mu_i - n)} \tag{99}$$

When μ consists only in one part N, this formula can be rewritten

$$q_n^\perp h_N = (\gamma N - \gamma n + 1) h_{N-n} \tag{100}$$

Since $q_n^\perp$ is a a derivation, one has its action on the multiplicative basis (h_μ), and one can use (100) to fully explicit it. We have $q_n^\perp = D_n + E_n$, where D_n and E_n are the derivations defined by

$$D_n h_n = h_{N-n} \tag{101}$$

and

$$E_n h_n = \gamma(N-n)h_{N-n}, \tag{102}$$

that is,

$$D_n = \sum_{r\geq 0} h_r \frac{\partial}{\partial h_{n+r}} \tag{103}$$

and

$$E_n = \gamma \sum_{r\geq 0} r h_r \frac{\partial}{\partial h_{n+r}} \tag{104}$$

so that

$$q_n^{\perp} = \sum_{r\geq 0} (1+\gamma r) h_r \frac{\partial}{\partial h_{n+r}} \tag{105}$$

We can see that this is valid also in the degenerate case $\gamma = 0$.

4.6. **Action of $q_n^{\perp}$ on the p_μ.** From (100), we have

$$q_n^{\perp} h_N = p_n^{\perp} h_N + \gamma(N-n)h_{N-n} \tag{106}$$

On another hand,

$$(N-n)h_{N-n} = p_1 h_{N-n-1} + p_2 h_{N-n-2} + p_3 h_{N-n-3} + \dots \tag{107}$$

Then,

$$\begin{aligned} q_n^{\perp} h_N &= p_n^{\perp} h_N + \gamma p_1 h_{N-n-1} + \gamma p_2 h_{N-n-2} + \dots \\ &= p_n^{\perp} h_N + \gamma p_1 (p_{n+1}^{\perp} h_N) + \gamma p_2 (p_{n+2}^{\perp} h_N) + \dots \\ &= p_n^{\perp} h_N + \gamma \sum_{r>n} p_{r-n} p_r^{\perp} h_N \end{aligned} \tag{108}$$

so that

$$q_n^{\perp} = p_n^{\perp} + \gamma \sum_{r>n} p_{r-n} p_r^{\perp} \tag{109}$$

5. A deformation of the Farahat-Higman algebra

5.1. **Recurrences for the structure constants of $\mathcal{H}'_\gamma$.** Denote by $a_{\lambda,\mu}^{\nu}(\gamma)$ the structure constants in the basis b_μ.

$$b_\lambda \star_\gamma b_\mu = \sum_{\nu} a_{\lambda,\mu}^{\nu}(\gamma) b_\nu \tag{110}$$

When $\gamma = 0$, the b_μ are identified with the g_μ, and then $a_{\lambda,\mu}^{\nu}(0)$ coincides with $a_{\lambda,\mu}^{\nu}$, the top connection coefficient.

Since $b_n = -q_n$, one has

$$a_{\lambda,(n)}^{\nu}(\gamma) = \langle b_\lambda \star_\gamma b_n, h_\nu^{\star}\rangle = -\langle b_\lambda \star_\gamma q_n, h_\nu^{\star}\rangle \tag{111}$$

Hence,

$$a^{\nu}_{\lambda,(n)}(\gamma) = -\langle b_\lambda, q_n^\perp h^\star_\nu\rangle \tag{112}$$

Since $q_n^\perp$ is a derivation, one can rewrite this as

$$a^{\nu}_{\lambda,(n)}(\gamma) = -\sum_i \langle b_\lambda, h^\star_{\nu\setminus(\nu_i)} q_n^\perp(h^*_{\nu_i})\rangle \tag{113}$$

The ith term in this sum corresponds to the coefficient of h^*_λ in

$$h^\star_{\nu\setminus(\nu_i)} q_n^\perp(h^*_{\nu_i}) \tag{114}$$

Hence this term gives a nonegative contribution only if there exists a partition μ such that

$$(\nu \setminus (\nu_i)) \cup \mu = \lambda, \tag{115}$$

that is,

$$\mu \cup \nu = \lambda \cup (\nu_i) \tag{116}$$

Hence, the ith term is also equal to the coefficient of h^*_μ in $q_n^\perp(h^*_{\nu_i})$, that is, $a^{\nu_i}_{\mu(n)}(\gamma)$. Summarizing, we have proved:

Theorem 5.1. *The structure constants $a^{\nu}_{\lambda,(n)}(\gamma)$ satisfy the recursion*

$$a^{\nu}_{\lambda,(n)}(\gamma) = \sum a^{\nu_i}_{\mu(n)}(\gamma) \tag{117}$$

where the sum is over the (i, μ) such that

$$\mu \cup \nu = \lambda \cup (\nu_i) \tag{118}$$

This formula is a generalization of (36), which is recovered for $\gamma = 0$.

5.2. **Multiplicative structure of the deformed Farahat-Higman algebra.** In the case where ν has only one part, there is a closed formula for $a^{\nu}_{\lambda(r)}(\gamma)$.

Theorem 5.2. *For $r \in \mathbb{N}$, $N \in \mathbb{N}$ and μ a partition,*

$$a^{(N)}_{\lambda,(r)}(\gamma) = \left(1 - \frac{N-r}{N+1}\gamma\right) a^{(N)}_{\lambda,(r)}(0) \tag{119}$$

$a^{(N)}_{\lambda,(r)}(0)$ corresponds to $a^{(N)}_{\lambda,(r)}$ in formula (35), where m is replaced by N.

Together with the recurrence formula (117), this determines completely the multiplicative structure of $\mathcal{H}'_\gamma$, since it is generated by the b_n. In order to derive (119), we shall need the following lemmas.

Lemma 5.3. *Let r and n be two positive integers, with $r < n$. Then,*

$$p_r^\perp h^\star_n = \sum_{\rho\vdash n-r} (-1)^{l(\rho)-1}(n+1)^{l(\rho)} \frac{p_\rho}{z_\rho} \tag{120}$$

Proof. From (75) we have

$$p_r^\perp h_n^\star = \sum_{\mu\vdash n} \frac{(-1)^{l(\mu)}(n+1)^{l(\mu)-1}}{z_\mu} p_r^\perp(p_\mu) \tag{121}$$

The action of $p_r^\perp$ on p_μ is

$$p_r^\perp(p_\mu) = r\frac{\partial p_\mu}{\partial p_r} = rm_r(\mu)p_{\mu\setminus(r)} \tag{122}$$

On another hand,

$$z_{\mu\setminus(r)} = \frac{z_\mu m_r(\mu\setminus(r))!}{rm_r(\mu)!} \tag{123}$$

with $m_r(\mu\setminus(r)) = m_r(\mu)-1$, hence $\frac{m_r(\mu)!}{m_r(\mu\setminus(r))!} = m_r(\mu)$, so that

$$\frac{z_\mu}{rm_r(\mu)} = z_{\mu\setminus(r)} \tag{124}$$

Hence,

$$\frac{p_r^\perp(p_\mu)}{z_\mu} = \frac{p_{\mu\setminus(r)}}{z_{\mu\setminus(r)}} \tag{125}$$

and

$$p_r^\perp h_n^\star = \sum_{\mu\vdash n}(-1)^{l(\mu)}(n+1)^{l(\mu)-1}\frac{p_{\mu\setminus(r)}}{z_{\mu\setminus(r)}} \tag{126}$$

or, equivalently,

$$p_r^\perp h_n^\star = \sum_{\rho\vdash n-r}(-1)^{l(\rho\cup(r))}(n+1)^{l(\rho\cup(r))-1}\frac{p_\rho}{z_\rho} \tag{127}$$

Since $l(\rho\cup(r)) = l(\rho)-1$, we deduce the required formula . ∎

Lemma 5.4. *Let r and N be two positive integers, with $N > r$. Then,*

$$\sum_{n>r} p_{n-r}(p_n^\perp h_N^\star) = -\frac{N-r}{N+1}p_r^\perp h_N^\star \tag{128}$$

Proof. From (120), one has

$$\sum_{n>r} p_{n-r}(p_n^\perp h_N^\star) = \sum_{n>r}\sum_{\rho\vdash n-r}(-1)^{l(\rho)-1}(n+1)^{l(\rho)}\frac{p_{\rho\cup(n-r)}}{z_\rho} \tag{129}$$

for $\mu = \rho\cup(n-r)$ and $k = n-r$, we obtain :

$$\sum_{n>r} p_{n-r}(p_n^\perp h_N^\star) = \sum_{\mu\vdash N-r}\sum_{k\in\mu}(-1)^{l(\mu)}(N+1)^{l(\mu)-1}\frac{p_\mu}{z_{\mu\setminus(k)}} \tag{130}$$

$$\sum_{n>r} p_{n-r}(p_n^\perp h_N^\star) = \sum_{\mu\vdash N-r}(-1)^{l(\mu)}(N+1)^{l(\mu)-1}\left(\sum_{k\in\mu}\frac{1}{z_{\mu\setminus(k)}}\right)p_\mu \tag{131}$$

From (124), one has

$$\sum_{k\in\mu}\frac{1}{z_{\mu\setminus(k)}}=\sum_{k\in\mu}\frac{km_\mu(k)}{z_\mu} \tag{132}$$

Hence,

$$\sum_{k\in\mu}\frac{1}{z_{\mu\setminus(k)}}=\frac{1}{z_\mu}\sum_{k\in\mu}km_\mu(k)=\frac{|\mu|}{z_\mu} \tag{133}$$

So we can rewrite (131) as

$$\sum_{n>r}p_{n-r}(p_n^\perp h_N^\star)=\sum_{\mu\vdash N-r}(-1)^{l(\mu)}(N+1)^{l(\mu)-1}\frac{N-r}{z_\mu}p_\mu \tag{134}$$

$$\sum_{n>r}p_{n-r}(p_n^\perp h_N^\star)=\frac{N-r}{N+1}\sum_{\mu\vdash N-r}(-1)^{l(\mu)}(N+1)^{l(\mu)}\frac{p_\mu}{z_\mu} \tag{135}$$

From (120), we get the required formula. ■

Proof – (of Theorem 5.2) Since $b_r=-q_r$ and $g_r=-p_r$, one has

$$\langle b_\lambda \star_\gamma b_r, h_N^\star\rangle=-\langle b_\lambda, q_r^\perp h_N^\star\rangle \tag{136}$$

Then, from (109) and lemma 5.3,

$$\begin{aligned}\langle b_\lambda \star_\gamma b_r, h_N^\star\rangle &= -\langle b_\lambda, p_r^\perp h_N^\star\rangle-\gamma\langle b_\lambda,\sum_{n>r}p_{n-r}(p_n^\perp h_N^\star)\rangle\\ &= -\langle g_\lambda, p_r^\perp h_N^\star\rangle+\gamma\frac{N-r}{N+1}\langle b_\lambda, p_r^\perp h_N^\star\rangle\\ &= -\langle g_\lambda p_r, h_N^\star\rangle+\gamma\frac{N-r}{N+1}\langle g_\lambda, p_r^\perp h_N^\star\rangle\\ &= \langle g_\lambda g_r, h_N^\star\rangle-\gamma\frac{N-r}{N+1}\langle g_\lambda g_r, h_N^\star\rangle\\ &= \left(1-\gamma\frac{N-r}{N+1}\right)\langle g_\lambda g_r, h_N^\star\rangle\end{aligned} \tag{137}$$

From that, we deduce (119).

6. A deformation of the Witt algebra

6.1. A simpler multiplicative formula in $\mathcal{H}'_\gamma$.

Now let us expand $q_k \star_\gamma q_n$ on the q_μ. One has

$$q_k\star_\gamma q_n=\sum_\mu\frac{1}{z_\mu}\langle q_k\star_\gamma q_n, p_\mu\rangle q_\mu, \tag{138}$$

so that

$$q_k\star_\gamma q_n=\sum_\mu\frac{1}{z_\mu}\langle q_k, q_n^\perp(p_\mu)\rangle q_\mu \tag{139}$$

From (109), deduce that

$$q_n^\perp p_\mu = p_n^\perp p_\mu + \gamma \sum_{r>n} p_{r-n} p_r^\perp p_\mu \tag{140}$$

so that

$$\begin{aligned} q_n^\perp p_\mu &= p_n^\perp p_\mu + \gamma \sum_{r>n} r p_{r-n} \frac{\partial p_\mu}{\partial p_r} \\ &= p_n^\perp p_\mu + \gamma \sum_{r>n} r m_r(\mu) p_{\mu\setminus(r)\cup(r-n)} \end{aligned} \tag{141}$$

Hence, q_μ gives to $q_k \star_\gamma q_n$ a contribution to the factor of γ only if $\mu = (k+n)$. In this case we have

$$\langle q_k, q_n^\perp p_{k+n}\rangle = \gamma(k+n)k \tag{142}$$

Hence,

$$q_k \star_\gamma q_n = q_{(k,n)} + k\gamma q_{k+n} \tag{143}$$

this formula also completely determines the multiplicative structure of $\mathcal{H}'_\gamma$. In the case $\gamma = 1$, we can see that it is coherent with the result of [3].

6.2. **Lie algebra structure corresponding to $\mathcal{H}'_\gamma$.** Now, suppose that $\gamma \neq 0$. Since $\mathcal{H}'_\gamma$ is a connected cocommutative Hopf algebra, it is the universal enveloping algebra of a Lie algebra $\mathcal{L}_\gamma$. From (143), its bracket is determined by

$$[q_k, q_n]_\gamma = \gamma(k-n)q_{k+n}. \tag{144}$$

Denote by $d_{n,\gamma}$ the differential operator

$$d_{n,\gamma} = t^{1-n\gamma}\frac{d}{dt} \tag{145}$$

The $d_{n,\gamma}$ satisfy the same relation (144) as the q_n of $\mathcal{L}_\gamma$, so that $\mathcal{H}'_\gamma$ can be interpreted as a Lie algebra of differential operators: it is the Lie algebra generated by the $d_{n,\gamma}$. In the case $\gamma = 1$ one has $d_{n,1} = t^{1-n}\frac{d}{dt}$. These operators generate $\mathcal{L}_1$, that is called the *Witt algebra*. It is known that the universal enveloping algebra of the Witt algebra is the dual of the Fàa di Bruno algebra. Note also that from the commutativity of $\mathcal{H}'_0$, (144) is also valid in the degenerate case $\gamma = 0$.

References

[1] P. Biane, *Minimal factorizations of a Cycle and Central Multiplicative Functions on the Infinite Symmetric Group*, J. Combin. Theor. A **76** (1996), 97–212.

[2] S. Corteel, A. Goupil and G. Schaeffer, *Content evaluation and class symmetric functions*, Adv. Math. **188** (2004), no. 2, 315–336.

[3] H. Figueroa and J. M. Gracia-Bondia, *Combinatorial Hopf algebras in quantum field theory I*, Rev.Math.Phys. **17** (2005), 881–976.

[4] H. K. Farahat and G. Higman, *The centres of symmetric group rings*, Proc. Roy. Soc. London Ser. A **250** (1959), 212–221.

[5] L. Foissy, *Faà di Bruno subalgebras of the Hopf algebra of planar trees from combinatorial Dyson-Schwinger equations*, Adv. Math. **218** (2008), no. 1, 136–162.

[6] I. P. Goulden and D. M. Jackson, *Symmetric functions and Macdonald's result for top connexion coefficients in the symmetric group*, J. Algebra **166** (1994), no. 2, 364–378.

[7] I. P. Goulden and D. M. Jackson, *Connexion coefficients for the symmetric group, free products in operator algebras and random matrices*, in Fields Institute Communications: Free Probability Theory, D.-V. Voiculescu, Ed. Providence, RI: Amer. Math. Soc., 1997, vol. 12, pp. 105–125.

[8] A.-A.A. Jucys, *Symmetric polynomials and the center of the symmetric group ring*, Rep. Mathematical Phys. **5** (1974), no. 1, 107–112.

[9] A. Lascoux and J.-Y. Thibon, *Vertex operators and the class algebras of symmetric groups*, Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) **283** (2001), Teor. Predst. Din. Sist. Komb. i Algoritm. Metody. 6, 156–177, 261; also in J. Math. Sci. (N. Y.) **121** (2004), no. 3, 2380–2392

[10] I.G. Macdonald, *Symmetric functions and Hall polynomials*, 2nd ed., Oxford University Press, 1995.

[11] S. Matsumoto and J. Novak, *Jucys-Murphy Elements and Unitary Matrix Integrals*, arXiv:0905.1992v2.

[12] J. Murray, *Generators for the centre of the group algebra of a symmetric group*, J. Algebra **271** (2004), 725–748.

Institut Gaspard Monge, Université Paris-Est Marne-la-Vallée, 5 Boulevard Descartes, Champs-sur-Marne, 77454 Marne-la-Vallée cedex 2, France

E-mail address: bultel@univ-mlv.fr